Transition from Continuous to Jumping Solutions in Quasi-static Elastic Contact Problems with Coulomb Friction: the Prediction of the Onset of Brake Squeal

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I will revisit the classical quasi-static elastic contact problem with Coulomb friction from a mathematical perspective (that is, existence and qualitative properties of solutions).

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set and $\partial\Omega = \Gamma_U \cup \Gamma_T \cup \Gamma_C$ a partition of its boundary into three disjoint parts (Dirichlet, Neumann and contact). The outward unit normal is denoted by $\mathbf{n}$. The framework is that of linear elasticity, so that the stress $\boldsymbol{\sigma}(\mathbf{u})$ is a linear function of the displacement $\mathbf{u}$. The boundary traction $\mathbf{t} := \boldsymbol{\sigma}(\mathbf{u})\,\mathbf{n}$ can be split into normal and tangential parts $\mathbf{t} = t_\mathrm{n}\mathbf{n} + \mathbf{t}_\mathrm{t}$ with respect to the reference configuration Ω. Considering the time interval $s \in [0, S]$, we are given time-varying body forces $\mathbf{F}(s) : \Omega \to \mathbb{R}^N$ and surface forces $\mathbf{T}(s) : \Gamma_T \to \mathbb{R}^N$. The quasi-static elastic contact problem with Coulomb friction is formally that of finding a time-varying displacement $\mathbf{u}(s) : \Omega \to \mathbb{R}^N$ satisfying a given initial condition and, for all $s \in [0, S]$:

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{F} = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_U, \\ \boldsymbol{\sigma}(\mathbf{u})\,\mathbf{n} = \mathbf{T}, & \text{on } \Gamma_T, \\ u_\mathrm{n} - g \le 0, \qquad t_\mathrm{n} \le 0, \qquad t_\mathrm{n}\,(u_\mathrm{n} - g) = 0, & \text{on } \Gamma_C, \\ \forall \mathbf{v} \in \mathbb{R}^N, \qquad \mathbf{t}_\mathrm{t} \cdot (\mathbf{v} - \dot{\mathbf{u}}_\mathrm{t}) - f t_\mathrm{n}(|\mathbf{v}| - |\dot{\mathbf{u}}_\mathrm{t}|) \ge 0, & \text{on } \Gamma_C. \end{cases}$$

Here, $\dot{\mathbf{u}}$ stands as usual for the time derivative of $\mathbf{u}$. The first line of the conditions on Γ_C are the Signorini conditions for unilateral contact with a rigid obstacle whose shape is encoded by the given gap function $g : \Gamma_C \to \mathbb{R}$. The second line of the conditions on Γ_C is a synthetic formulation of the ubiquitous Coulomb law of dry friction, with given friction coefficient $f \ge 0$.

The quasi-static elastic contact problem with Coulomb friction is invariant under monotone reparameterization of time. It therefore falls into the class of *rate-independent processes*, such as perfect elastoplasticity [3] or brittle fracture [4]. A natural mathematical framework for the analysis of rate-independent processes is that of functions with *bounded variation in time*, both for the solutions and the loads. Such functions may have jump discontinuities in time. Therefore, a natural question to ask about rate-independent processes is whether solutions can have spontaneous jumps in time, even when the loads are smooth: do absolutely continuous loads always yield absolutely continuous solutions, or not?

- In the case of brittle fracture, finite jumps (in time) in the solution produced by infinitesimal changes in the load have been evidenced in the quasi-static theory. This was interpreted as the fact that unstable crack propagation makes brittle fracture inherently a dynamic process, which should be analyzed in the framework of elastodynamics. Besides, this is evidenced by the sound emitted by crack propagation in glass.

- In the case of perfect elastoplasticity, absolutely continuous loads always yield absolutely continuous solutions [3]. Besides, it is common experience that the plastic bending of a metallic spoon is silent.

After having provided the appropriate formulation of the quasi-static elastic contact problem with Coulomb friction in the framework of functions with bounded variation in time,

- I will present examples (large friction coefficients) of solutions with spontaneous jumps in time, while no continuous-in-time solution exists, despite absolutely continuous loads,
- I will present a theorem (only in 2D, for now) ensuring that in the case of small friction coefficients, absolutely continuous loads always yield absolutely continuous solutions,
- I will present the optimal condition on the friction coefficient separating these two regimes.

These results extend the early analysis of Klarbring [5] of the two-degrees-of-freedom problem where the elasticity operator is replaced by the 2×2 positive definite symmetric stiffness matrix:

$$\mathbf{K} = \begin{pmatrix} k_{\mathrm{nn}} & k_{\mathrm{nt}} \\ k_{\mathrm{nt}} & k_{\mathrm{tt}} \end{pmatrix}.$$

Klarbring studied existence and uniqueness of solution for the *rate problem* (given a current configuration at time s and a velocity of the loads, find the velocity of the solution at time s). He showed that:

- if $f|k_{\mathrm{nt}}| < k_{\mathrm{tt}}$ (Klarbring's condition), then the rate problem always has a unique solution,
- if $f|k_{\mathrm{nt}}| \geq k_{\mathrm{tt}}$, then the rate problem can have either no solution or several solutions.

Formulating Klarbring's problem in the bounded variation framework, I will show that the two-degrees-of-freedom quasi-static problem with arbitrarily large friction coefficient always has solutions with bounded variation, possibly with spontaneous jumps in time (in the cases where Klarbring's rate problem has no solution). I will also prove that, under Klarbring's condition $f|k_{\mathrm{nt}}| < k_{\mathrm{tt}}$, absolutely continuous loads always yield absolutely continuous solutions (no spontaneous jumps in time, if the loads have no jumps). Hence, the new results I will present can be seen as a generalization of Klarbring's condition to the case of a continuum, paving the road to practical design against the onset of friction-induced vibrations, such as brake squeal.

References

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