Finite Element Methods for Elastic Contact: Augmented Lagrangian and Nitsche

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We consider three alternatives for solving elastic contact problems. The first one is a stable mixed method. For the Lagrange multiplier we either use the trace space of the field variable or a discontinuous piecewise polynomial. The first choice yields a stable method, cf., e.g., [1]. In the second one, the field value has to be augmented with bubble degrees of freedom to obtain stability [2]. For the latter choice the Augmented Lagrangian method has been proved to be very efficient to solve the linear system that arise. That applies also for the first choice, but due to the continuous Lagrange multiplier, the band-with of the penalty term is greater than that of the stiffness matrix from the field variable. In [3] it is shown that a bigger penalty term yields a faster convergence. There is however a trade-off with the conditioning of the system to be solved at each iteration. The requirement that this conditioning should be that of second order elliptic equations, leads to the choice $O(h^{-1})$ for the penalty parameter.

The second choice is to choose the mixed method with continuous Lagrange multiplier but based on the Augmented Lagrangian formulation of the continuous system, not the discrete. This choice appears to be in common use, but we have not found an analysis. By adapting the arguments in [3] we prove the convergence. There is, however, a fundamental difference between this and the first choice, if the penalty parameter is chosen too large, the convergence rate of the discretization is reduced. The analysis shows that the optimal choice is again $O(h^{-1})$.

This leads us to the third method, that of J.A. Nitsche. With the choice $O(h^{-1})$, the second choice resembles an unsymmetric Nitsche method. We introduce Nitsche in this spirit by adding a symmetrizing term. The final method is symmetric, optimally conditioned, and solved in one shot, no iteration needed.

References

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